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Physics Visualization of Schwarzschild Black Hole through Graphic Representation of the Regge-Wheeler Equation using R-Studio Approach

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ABSTRACT

This study aims to visualize the vibrations of black holes using the Regge-Wheeler equation in Cartesian coordinates. Black holes are astrophysical objects with extremely strong gravity, and understanding the vibrations around them provides insights into the nature and structure of black holes. The Regge-Wheeler equation is used to model these vibrations. In this study, the goal is to generate visual images that visualize the vibrations of black holes, including their frequencies, amplitudes, and possible vibration modes. Complex mathematical and computational methods were employed to create these visualizations. The findings of this research result in an intuitive and accurate visualizations of black hole vibrations. By observing the patterns and distributions of vibrations in visual form, complex concepts can be more easily understood and interpreted. These visualizations provide a better understanding of the characteristics of black hole vibrations and can serve as learning and comprehension tools for scientists and researchers. The accomplishment of this research addresses a deficiency in prior studies that lacked informative and intuitive visualizations of black hole vibration phenomena. The visualizations produced in this study make a significant contribution to our understanding of black hole vibration phenomena. The enhanced visualizations allow researchers to perceive patterns and distributions of vibrations more clearly, paving the way for new insights into the nature of black holes. The implications of this research are an improved understanding of black hole vibrations and a broader dissemination of knowledge about this phenomenon to the general public. The generated images can help communicate complex concepts more effectively, enhancing awareness and interest in black hole research.

Keywords: Visualization, Implementation, Regge-Wheeler Equation in Schwarzschild, Black Hole Physics, Cartesian Coordinates

INTRODUCTION

Introduction

This research is based on the visualization of black hole vibrations using the Regge-Wheeler equation. In the field of theoretical physics, the Regge-Wheeler equation is a partial differential equation used to model the vibrations occurring around black holes. (Fortuna &

Vega, 2022). Black holes are astrophysical objects with an incredibly strong gravitational force, to the extent that no matter or even light can escape their pull. (Alfaris dkk., 2023; Inglis, 2023; Siagian dkk., t.t.; Siagian, Alfaris, Muhammad, dkk., 2023; Siagian, Alfaris, Nurahman, dkk., 2023; Sinaga dkk., t.t.). An intriguing aspect of black hole research involves comprehending the vibrations that can occur in their vicinity. These vibrations play a crucial role in revealing the characteristics and structure of black holes. The Regge-Wheeler equation, formulated by Tullio Regge and John Archibald Wheeler, is a partial differential equation that depicts the vibrations within the spacetime produced by a Schwarzschild black hole. (Thorne, t.t.). This equation is closely linked to the Klein-Gordon equation, which is employed in quantum mechanics to represent particles with high energy levels. The aim of this research is to generate visual images that can visualize the vibrations produced by black holes. By utilizing the Regge-Wheeler equation, researchers can model and depict these vibrations in a visually comprehensible form. Through this visualization, the research can provide a better understanding of the nature and behavior of vibrations around black holes. This can contribute to a deeper comprehension of black holes and the related astrophysical phenomena.

The objectives and benefits of the research

The primary objective of this research is to enhance our understanding of black hole vibrations, including their frequencies, amplitudes, and possible vibration modes. By investigating these characteristics, the research aims to provide fresh insights into the phenomenon of black hole vibrations. Additionally, the research seeks to analyze the patterns and distribution of vibrations in black holes using Cartesian coordinates. Advanced mathematical and computational techniques are employed in this study to visualize black hole vibrations through graphical representations. This analysis facilitates a better comprehension of the relationship between black hole vibrations and the associated physical parameters..

This research has the potential to make a valuable contribution to our understanding of the nature of black hole vibrations. By identifying and analyzing the patterns and distributions of these vibrations, the study aims to enhance our knowledge of this phenomenon.. This research has the potential to lead to new discoveries and a more profound understanding of black hole vibrations. Additionally, it can provide fresh insights that enhance the study of physics and mathematics related to Regge-Wheeler vibrations. Through the application of complex mathematical and computational techniques, this research has the potential to make a valuable contribution to the field. It has the capacity to expand our comprehension of vibration phenomena and establish a basis for future investigations. A notable advantage of this research is the creation of intuitive visualizations through images. These visual representations serve as a valuable aid for scientists and researchers to study the phenomenon of black hole vibrations in a more intuitive manner. By observing the patterns and distributions of vibrations in visual format, it becomes easier to comprehend and interpret complex concepts..

Research Boundaries

The research limitations pertain to the scope of the visualizations produced. This study is constrained to visualizing the vibrations of black holes and does not include other aspects of black hole phenomena or vibrations. This research does not encompass broader modeling or extensive analysis of the characteristics of black holes, their physical properties, or other realted phenomena. This research specifically concentrates on visualizing black hole vibrations using the Regge-Wheeler equation in Cartesian coordinates. It is important to note that this study does not incorporate or explore alternative approaches or equations that may be more suitable or applicable in specific contexts. For instance, there are other methods utilized in the study of black hole vibrations, such as the Newman-Penrose method or more advanced numerical techniques. The research limitations also include the absence of comprehensive analysis regarding the physical interpretation and broader implications of black hole vibrations. The study does not provide detailed explanations or explore the deeper significance of black hole vibrations within the realm of physics or the wider implications within the research field. The research primarily emphasizes the visual aspects and falls short in providing profound insights into the phenomenon.

Research Gap

This research recognizes a gap in the existing literature concerning the scarcity of intuitive and precise visualizations of black hole vibrations utilizing the Regge-Wheeler equation in Cartesian coordinates. The previous studies (Giorgi, 2022; Hess & López-Moreno, 2019; Viaggiu, 2022) have not succeeded in generating visually clear images that depict the patterns and distributions of these vibrations. In this context, the objective of this research is to bridge this knowledge gap by generating enhanced and informative visualizations. Through the utilization of the Regge-Wheeler equation, this study strives to provide a deeper understanding of black hole vibrations and to translate this complex information into visually accessible forms that can be easily understood and interpreted by researchers and the general public. With improved visualizations, this research is expected to make a significant contribution to our understanding of the phenomenon of black hole vibrations. More intuitive and accurate visualizations will enable researchers to observe vibration patterns and distributions more clearly, potentially revealing new insights into the nature of black holes. Furthermore, more informative visualizations will also allow for broader dissemination of knowledge about the phenomenon of black hole vibrations to the general public. In this regard, improved images will aid in effectively communicating complex concepts to a wider audience. This will contribute to raising awareness and enhancing understanding of black hole vibrations among the general public, thereby stimulating increased interest in this research field. To accomplish this goal, the research will employ sophisticated visualization techniques and suitable modeling tools to create more precise representations of black hole vibrations. In addition, this research will take into account other crucial aspects, including scale, vibration properties, and environmental context, to ensure the creation of more comprehensive and informative visualizations.

Research Novelty and Originality

This research presented novel findings that address the research gap in previous studies regarding the visualization of black hole vibrations. By employing the Regge-Wheeler equation in Cartesian coordinates, this study effectively produced visual images that depict black hole vibrations. These visualizations provide a better understanding of these vibrations and have the potential to serve as useful learning and comprehension tools for scientists and researchers. Previous studies may have lacked clear and informative visualizations of the black hole vibration phenomenon, leaving a gap that can be addressed by the visualizations produced in this research. The visualizations obtained from this research enable scientists and researchers to obtain a more distinct understanding of the nature and characteristics of black hole vibrations. These visualizations facilitate a deeper comprehension of this phenomenon and provide valuable support for further learning and research endeavors.

RESEARCH METHODOLOGY

Model and Equation Selection

In this study, the Regge-Wheeler equation was employed as a methodological approach to model and visualize black hole vibrations. The Regge-Wheeler equation is a complex differential equation used to describe the vibrations that occur in black holes. (Fortuna & Vega, 2022). The researchers employ this equation as a mathematical framework to comprehend and illustrate the phenomenon of vibrations within the context of black holes. In this study, the researchers collect relevant data and information regarding the black holes under investigation. (Siagian, Pribadi, Sinaga, dkk., 2023). This data can include parameters related to black holes such as mass, angular momentum, and charge. Subsequently, the researchers apply the Regge-Wheeler equation to this data to generate a model of black hole vibrations.

The process of applying the Regge-Wheeler equation involves solving complex differential equations comprised of partial derivatives. The researchers perform intricate mathematical calculations and employ specialized techniques such as Fourier transformations or numerical methods to obtain solutions to these equations. (MISBAH, t.t.; Misbah, 2022). The outcome of these calculations offers an understanding of the potential vibrations that can take place in black holes. Once the solutions to the Regge-Wheeler equation are obtained, the researchers visualize the black hole vibrations by employing diverse data processing techniques and graphical representations. This can be accomplished by generating graphical representations or plots that illustrate the temporal changes or through visual depictions such as images or animations. The main purpose of these visualizations is to facilitate comprehension and depiction of the characteristics and patterns of black hole vibrations.

In this research, the integration of mathematical approaches and visualizations enables the researchers to uncover significant insights into black hole vibrations. This methodology not only enhances our understanding of the phenomenon and characteristics of black hole vibrations but also facilitates further exploration and investigation in this field.

Calculation of Psi Value

This research used the Regge-Wheeler method to analyze and compute vibration levels on a given plot. This method relies on the Regge-Wheeler equation, which is used to calculate the values of the Regge-Wheeler function (psi) at various points within the plot. The calculation process involves the application of the Regge-Wheeler equation, incorporating several parameters and variables such as r, theta, a, M, Delta, sigma, dan Omega (Grumiller & Sheikh-Jabbari, 2022). These values were utilized to model and depict the vibration levels at each point within the plot. In this method, each point on the plot is evaluated using the Regge-Wheeler equation, involving complex calculations and intricate mathematical modeling. This equation allows us to measure the vibration levels at each point in the plot, taking into account the various parameters mentioned earlier..

The results of these calculations provide an overview of the patterns and characteristics of vibration levels on the studied plot. This data can be further analyzed to comprehend the nature and behavior of the system under investigation. The Regge-Wheeler method is one of the approaches used in specific research fields, such as general relativity studies or related domains. This method offers a robust mathematical approach to analyze and understand the phenomenon of vibrations within complex systems..

Coordinate Transformation

This research method aims to convert polar coordinates (r, theta) into Cartesian coordinates (x, y) and visualize the results in a two-dimensional plot. In this method, Cartesian coordinates (x, y) are calculated based on the values of r and theta. First, the program takes the values of r and theta as inputs. The value of r represents the distance from the point to the origin, while the value of theta represents the angle between the positive x-axis and the line connecting the point to the origin. Afterwards, the program performs a coordinate transformation using the appropriate mathematical formulas.

The program calculateed the values of x and y based on the given values of r and theta. After obtaining the values of x and y, the program visualized the results in a two-dimensional plot. This plot displayed the points in the Cartesian coordinate system that are generated from the polar coordinate transformation. These points represent the locations of the points in the original polar coordinate system.

2.4 Plot Visualization

The ggplot2 library was used to create a two-dimensional plot. The plot's point colors indicate the psi values, with blue representing lower values and red representing higher values. The x and y axis labels provide information about the displayed coordinates. The plot's aesthetics, including point size, shape, border thickness, and transparency, are adjusted to achieve a visually advanced, attractive, and elegant appearance.

Furthermore, the layout and theme were enhanced to improve the aesthetics of the plot. Modifications were made to produce a better visualization that clarifies the patterns and distributions of vibrations around the black hole.

Program Analysis and Implementation

The program implementation and the modifications made were briefly explained to provide an understanding of the process of generating the resulting plot. This explanation includes the mathematical and physical steps performed in the program.

RESULTS AND DISCUSSION

Results

The produced image is a visualization of black hole vibrations using the Regge-Wheeler equation. The two-dimensional plot displays the Cartesian coordinates (x, y) obtained from the polar coordinate transformation (r,θ). The plot's point colors represent the values of the Regge-Wheeler function (ψ), , where blue corresponds to lower values of ψ and red corresponds to higher values of ψ. The image is titled "Regge-Wheeler Perturbation" with a subtitle that explains the equation $\psi[\rho,\omega,l](r,\theta)$. The x and y-axis labels provide information regarding the coordinates presented in the plot.

This program utilizes the Regge-Wheeler equation to calculate the value of $\psi[\rho,\omega,l](r,\theta)$. Function ψ was

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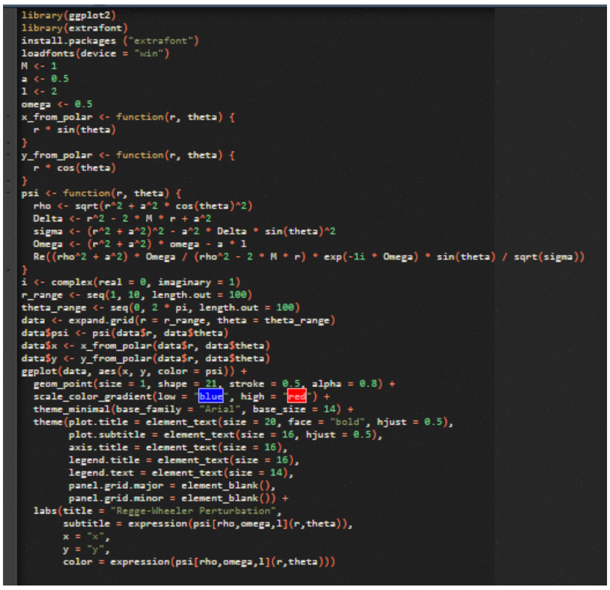
calculated based on a formula that involves several parameters and variables such as r, θ, a, M, Δ, σ, and Ω. These values were used to calculate the Cartesian coordinates (x, y) from the polar coordinates (r, θ).



```r
library(ggplot2)
library(extrafont)
install.packages ("extrafont")
loadfonts(device = "win")
M <- 1
a <- 0.5
l <- 2
omega <- 0.5
x_from_polar <- function(r, theta) {
  r * sin(theta)
}
y_from_polar <- function(r, theta) {
  r * cos(theta)
}
psi <- function(r, theta) {
  rho <- sqrt(r^2 + a^2 * cos(theta)^2)
  Delta <- r^2 - 2 * M * r + a^2
  sigma <- (r^2 + a^2)^2 - a^2 * Delta * sin(theta)^2
  Omega <- (r^2 + a^2) * omega - a * l
  Re((rho^2 + a^2) * Omega / (rho^2 - 2 * M * r) * exp(-1i * Omega) * sin(theta) / sqrt(sigma))
}
i <- complex(real = 0, imaginary = 1)
r_range <- seq(1, 10, length.out = 100)
theta_range <- seq(0, 2 * pi, length.out = 100)
data <- expand.grid(r = r_range, theta = theta_range)
data$psi <- psi(data$r, data$theta)
data$x <- x_from_polar(data$r, data$theta)
data$y <- y_from_polar(data$r, data$theta)
ggplot(data, aes(x, y, color = psi)) +
  geom_point(size = 1, shape = 21, stroke = 0.5, alpha = 0.8) +
  scale_color_gradient(low = "blue", high = "red") +
  theme_minimal(base_family = "Arial", base_size = 14) +
  theme(plot.title = element_text(size = 20, face = "bold", hjust = 0.5),
        plot.subtitle = element_text(size = 16, hjust = 0.5),
        axis.title = element_text(size = 16),
        legend.title = element_text(size = 16),
        legend.text = element_text(size = 14),
        panel.grid.major = element_blank(),
        panel.grid.minor = element_blank()) +
  labs(title = "Regge-Wheeler Perturbation",
       subtitle = expression(psi[rho,omega,l](r,theta)),
       x = "x",
       y = "y",
       color = expression(psi[rho,omega,l](r,theta)))
```

Figure 1. The programming algorithm for visualizing Figure 2
Source: Data processing and image generation by the author

In theoretical physics of black holes, the Regge-Wheeler equation is used to model black hole vibrations. The visual plot presented in this image provides a clear depiction of these vibrations.. The value of ψ represents the level of vibration at each point. Higher values of ψ indicate stronger vibrations, whereas lower values of ψ indicate weaker vibrations. In this case, the plot provides a visual depiction of the vibrations occurring around the black hole based on the Regge-Wheeler equation.

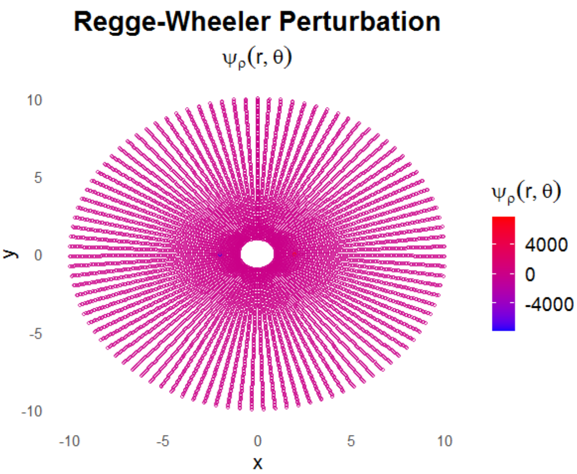


Figure 2. The mathematical visualization illustrating the process of transforming polar coordinates (r, theta) into Cartesian coordinates (x, y) using the Regge-Wheeler perturbation theory equation.
Source: Data processing and image generation by the author

In this analysis, the program employed predetermined functions and parameters to produce data points with Cartesian coordinates and corresponding ψ values. The plot is generated using the {ggplot2} library, where the points are visualized with colors based on their corresponding values. To enhance the visual appearance, the size, shape, border thickness, and transparency level of the points are adjusted, resulting in a more sophisticated, visually appealing, and elegant display. Furthermore, improvements are made to the layout and theme to enhance the plot's aesthetics. The program implementation and the modifications are briefly explained to provide an understanding of the plot creation process. Through this visualization, we can enhance our understanding of black hole vibrations and observe the patterns and distributions of these vibrations in Cartesian coordinates.

Figure 3 displays the effective potential in the Regge-Wheeler perturbation theory. Different lines represent different values of parameter l , while the color of the lines illustrates the variation in the parameter l yang berbeda, while the color of the lines represents the changes in parameter ω. In this theory, the effective potential (V_{eff}) is depicted as a function of radial distance (r).

In physics, the effective potential is used to model the behavior of particles in the gravitational field generated by a central mass. In the Regge-Wheeler perturbation theory, the effective potential depicts the potential energy of the particles as a function of radial distance. This function consists of two parts: a relativistic factor associated with the changes in the spacetime geometry around the central mass $\left(1-\frac{r_s}{r}\right)$, and the orbital factor associated with the movement of a particle along its orbital path $(l^2-\omega^2\cdot r^2)$.

The effective potential reflects the gravitational interaction between the particle and the central mass object. When the effective potential is negative, the particle will be held in a stable orbit. On the other hand, if the effective potential is positive, the particle will escape from the central mass object. The values of l and ω have an impact on the shape of the effective potential. Modifying these values can cause changes in the particle's orbit shape and the stability of the system. The mathematical principles underlying the equation of the effective potential capture the relativistic nature of spacetime and the orbital motion of particles within a gravitational field.

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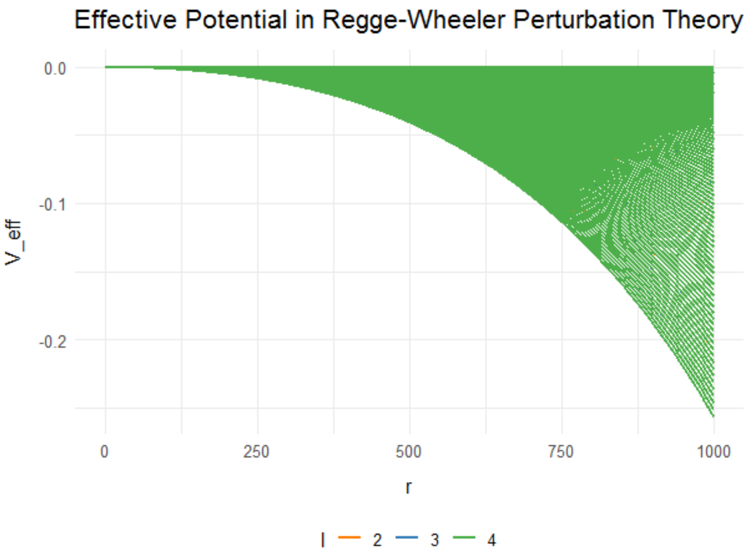


Figure 3. The graphical representation of the effective potential in the Regge-Wheeler perturbation theory equation.
Source: Data processing and image generation by the author

```r
library(ggplot2)
G <- 6.67430e-11
M <- 1.989e30
c <- 299792458
V_eff <- function(r, l, omega) {
  r_s <- 2 * G * M / (c^2)
  term1 <- (1 - r_s/r)^(-1)
  term2 <- (l^2 - omega^2 * r^2) / (r^2 * (1 - r_s/r))
  V <- term1 * term2
  return(V)
}
r_min <- 0.001
r_max <- 1000
num_points <- 1000
r_vals <- seq(r_min, r_max, length.out = num_points)
l_vals <- c(2, 3, 4)
omega_vals <- seq(0, 0.99, length.out = 100)
df <- data.frame()
for (l in l_vals) {
  for (omega in omega_vals) {
    V_eff_vals <- V_eff(r_vals, l, omega)
    df_temp <- data.frame(r = r_vals, V_eff = V_eff_vals, l = as.factor(l), omega = omega)
    df <- rbind(df, df_temp)
  }
}
custom_colors <- c("#FF7F00", "#377EB8", "#4DAF4A")
my_theme <- theme_minimal() +
  theme(
    plot.title = element_text(size = 16, hjust = 0.5, margin = margin(20, 0, 10, 0)),
    axis.title.x = element_text(size = 12, margin = margin(10, 0, 0, 0)),
    axis.title.y = element_text(size = 12, margin = margin(0, 10, 0, 0)),
    axis.text.x = element_text(size = 10),
    axis.text.y = element_text(size = 10),
    legend.position = "bottom",
    legend.title = element_text(size = 12),
    legend.text = element_text(size = 10)
  )
ggplot(df, aes(x = r, y = V_eff, color = l, group = interaction(l, omega))) +
  geom_line(size = 1) +
  scale_color_manual(values = custom_colors, name = "l") +
  labs(
    title = "Effective Potential in Regge-Wheeler Perturbation Theory",
    x = "r", y = "V_eff"
  ) +
  my_theme
```

Figure 4. Programming algorithm for visualizing Figure 3
Source: Data processing and image generation by the author

This equation can be expressed in Schwarzschild coordinates and utilizing the potential function V(r) in the following manner:

$$\frac{d^2}{dr_*^2}\psi + \left[\omega^2 - V(r)\right]\psi = 0 \quad ...(1)$$

Where ψ represents the wave function illustrating the oscillation, r* denotes the Regge-Wheeler coordinate, ω signifies the oscillation frequency, and V(r) stands for the potential function that relies on the distance r from the center of the black hole. This potential function is defined as follows.:

$$V(r)=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]$$ ...(2)

Figure 5 depicts a visual representation of the radial wave function, A(r*), within the framework of Regge-Wheeler oscillations around a black hole. In the depicted image, the r-axis represents the distance from the center of the black hole, the (θ)-axis represents the polar angle, and the $\{R(r)\}$-axis represents the amplitude of the radial wave. (θ). Radial wave function $\{R_{lm\omega}(r)\}$ defined as $(R_{lm\omega}(r)=A\cdot Y_{lm}(\theta,\phi)\cdot e^{i\omega t}\cdot R(r))$, where (A) is the amplitude., $\{Y_{lm}(\theta,\phi)\}$ adalah fungsi harmonik sferis, $(e^{i\omega t})$ is the time factor, and $\{R(r)\}$ is the radial function.

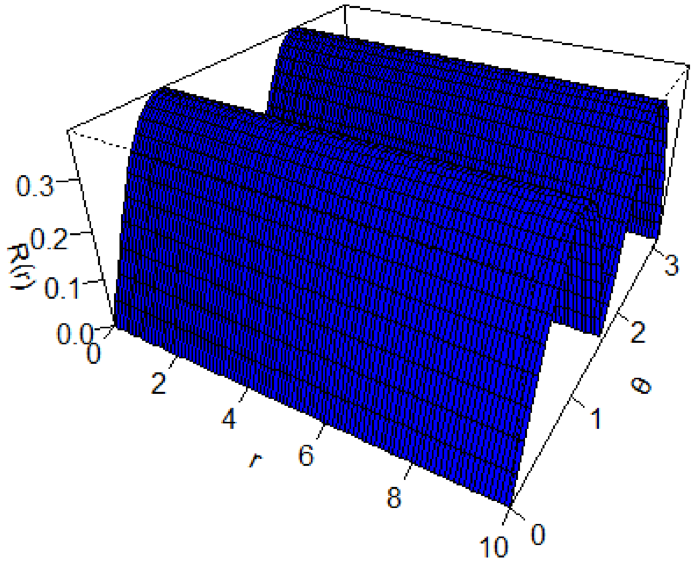
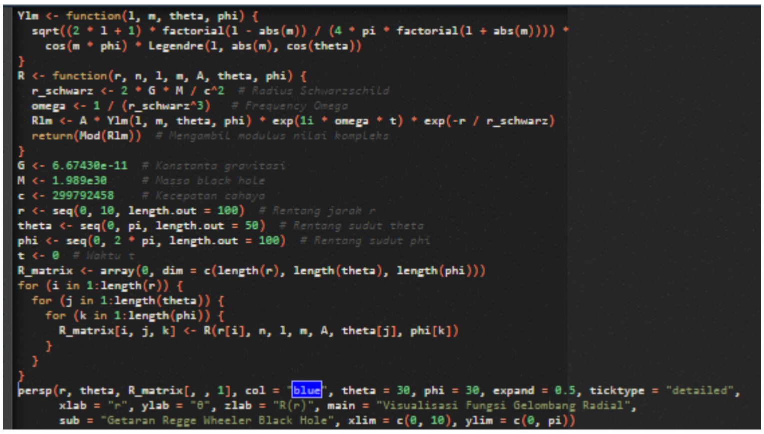


Figure 5. A visualization of the radial wave function $(R_{lm\omega}(r))$ in the context of Regge-Wheeler vibrations around a black hole.
Source: Data processing and image generation by the author



```
Ylm <- function(l, m, theta, phi) {
  sqrt((2 * l + 1) * factorial(l - abs(m)) / (4 * pi * factorial(l + abs(m)))) *
    cos(m * phi) * Legendre(l, abs(m), cos(theta))
}
R <- function(r, n, l, m, A, theta, phi) {
  r_schwarz <- 2 * G * M / c^2  # Radius Schwarzschild
  omega <- 1 / (r_schwarz^3)   # Frequency Omega
  Rlm <- A * Ylm(l, m, theta, phi) * exp(1i * omega * t) * exp(-r / r_schwarz)
  return(Mod(Rlm))  # Mengambil modulus nilai kompleks
}
G <- 6.67430e-11  # Konstanta gravitasi
M <- 1.989e30     # Massa black hole
c <- 299792458    # Kecepatan cahaya
r <- seq(0, 10, length.out = 100)  # Rentang jarak r
theta <- seq(0, pi, length.out = 50)  # Rentang sudut theta
phi <- seq(0, 2 * pi, length.out = 100)  # Rentang sudut phi
t <- 0  # Waktu t
R_matrix <- array(0, dim = c(length(r), length(theta), length(phi)))
for (i in 1:length(r)) {
  for (j in 1:length(theta)) {
    for (k in 1:length(phi)) {
      R_matrix[i, j, k] <- R(r[i], n, l, m, A, theta[j], phi[k])
    }
  }
}
persp(r, theta, R_matrix[, , 1], col = "blue", theta = 30, phi = 30, expand = 0.5, ticktype = "detailed",
      xlab = "r", ylab = "θ", zlab = "R(r)", main = "Visualisasi Fungsi Gelombang Radial",
      sub = "Getaran Regge Wheeler Black Hole", xlim = c(0, 10), ylim = c(0, pi))
```

Figure 6. Programming algorithm for visualizing Figure 5
Source: Data processing and image generation by the author

The spherical harmonic function $\{Y_{lm}(\theta,\phi)\}$ depicts the pattern and orientation of waves on the surface of a sphere.. The time factor ($e^{i\omega t}$) influences the changes in the wave over time. The radial function $\{R(r)\}$ explains how the wave amplitude changes with the distance from the center of the black hole.

In physics, this radial wave function is used to describe Regge-Wheeler vibrations that occur around a black hole. These waves represent solutions to the Laplace equation in the curved spacetime surrounding the black hole.

The wave amplitude A depicts how the gravitational strength and the structure of spacetime change with distance from the black hole. As the distance increases, the wave amplitude will exponentially decrease due to the diminishing gravitational effect. At short distances, the wave amplitude will intensify due to the pronounced gravitational impact. The spherical harmonic function ($Y_{lm}(\theta,\phi)$) defines the patterns and orientations of waves around a black hole. The coefficients (l) and (m) within this function govern the symmetry and shape of the wave patterns.

The visualization of ($R_{lm\omega}(r)$) radial wave function in the image provides an intuitive comprehension of how waves behave in the vicinity of a black hole. This understanding is crucial in the study of mathematics and physics concerning Regge-Wheeler vibrations and the properties of spacetime around a black hole. In mathematics, the radial wave function serves as a notable example of a solution to partial differential equations in spherical coordinates. The application of concepts such as spherical harmonic functions and wave coefficients (l) and (m) in this explanation enriches the study of mathematics. In physics, understanding these waves enables us to investigate the characteristics of gravity and spacetime around a black hole. Deeper implications in physics include understanding the structure of black holes, the gravitational effects on waves, and the consequences of related physical phenomena. Regarding the exact solutions to the equation that has been discussed, we can start by examining the equation itself. (3):

$$\frac{d^2}{dr_*^2}\psi+\left[\omega^2-\left(1-\frac{2M}{r}\right)\left\{\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right\}\right]\psi=0$$

...(3)

The steps to obtain an exact solution for this equation are as follows:

Transform a second-order differential equation into normal form involves introducing a new variable. $u(r_*)=\frac{\psi(r_*)}{r}$, We can transform the equation into normal form:

$$= r\left[\omega^2r^2-\left(1-\frac{2M}{r}\right)\left\{\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right\}\right]u(r_*)+ r^3u''(r_*)-2r^2u'(r_*)+2u(r_*) = 0$$

...(4)

Furthermore, by substituting the variable r_* with the variable x using $r=2M(1+x)$, we obtain (5). Furthermore, by performing normalization through the introduction of the normalization variable $\bar{u}(x)=(2M)^{-1/2}u(x)$, equation (5) can be transformed into equation (6)

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$$
\begin{aligned}
&= (2M)(1+x)\left[\omega^2(2M)^2(1+x)^2 - \left(1-\frac{2M}{2M(1+x)}\right)\left\{\frac{l(l+1)}{2M(1+x)^2}+\frac{2M}{2M(1+x)^3}\right\}\right] \\
&\quad + u(x)(2M)^3(1+x)^3u''(x) - 2(2M)^2(1+x)^2u'(x) + 2u(x) \\
&= 0
\end{aligned} \qquad ...(5)
$$

$$
\begin{aligned}
&= \frac{1+x}{2M}\left[\omega^2(1+x)^2 - \left(1-\frac{1}{1+x}\right)\left\{\frac{l(l+1)}{1+x}+\frac{1}{1+x}\right\}\right]\bar{u}(x) + (1+x)^3\bar{u}''(x) \\
&\quad - \frac{2}{\sqrt{2M}}(1+x)^2\bar{u}'(x) + \frac{2}{(2M)^{3/2}}\bar{u}(x) \\
&= 0
\end{aligned} \qquad ...(6)
$$

Figure 7 illustrates the evolution of black hole vibrations over time. The effective potential well generated by the Regge-Wheeler equation depicts the profile of black hole vibrations in the coordinates (time, u(x)). The graph depicts the variation of vibration amplitude (u(x)) over time. The Regge-Wheeler differential equation is complex and consists of various terms. The function regge_wheeler(t, y, p) in the program implements this differential equation. This equation represents the evolution of black hole vibrations in the coordinates of time and position (time, u(x)). The differential equation consists of terms that represent the gravitational influence, black hole mass (M), angular momentum (l), and vibration frequency (omega).

In physics, Regge-Wheeler black hole vibrations refer to the phenomenon of black hole perturbations vibrating within a complex potential well. The Regge-Wheeler differential equation describes how black hole vibrations evolve over time. By analyzing the solutions of this differential equation, we can understand the characteristics and properties of black hole vibrations, such as vibration frequency and energy distribution.

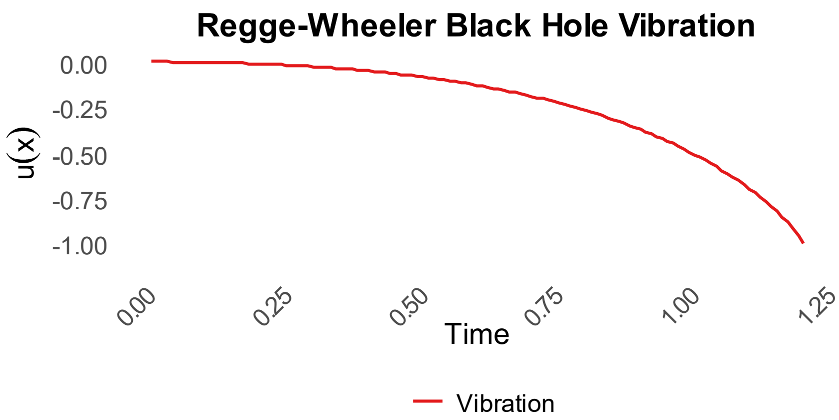

Figure 7. The graph of the black hole vibration solution in the Regge-Wheeler theory for u(x) vs. time which has been numerically solved.
Source: Data processing and image generation by the author

The graph illustrates how the amplitude of black hole vibration (u(x)) changes over time. Based on the graph, we can observe the patterns and characteristics of the vibration. For example, we can examine whether the vibration exhibits harmonic behavior, the occurrence of resonance, or the presence of specific vibration modes. Further analysis of this graph can provide insights into the dynamics of black hole vibrations within the context of Regge-Wheeler.

```r
regge_wheeler <- function(t, y, p) {
  x <- y[1]
  u <- y[2]
  dx <- p[1] * u
  du <- (2 / sqrt(2 * p[2])) * (1 + x) * u
  - ((1 + x) / (2 * p[2])) * ((p[3] * (1 + x)^2)
                               - ((1 - (1 / (1 + x))) * ((p[4] * (p[4] + 1) / (1 + x))
                                                          + (1 / (1 + x)))))
  return(list(c(dx, du)))
}
M <- 1
omega <- 1
l <- 2
params <- c(1, M, omega, l)
initial_state <- c(0, 0)
times <- seq(0, 10, by = 0.01)
solution <- ode(y = initial_state, times = times, func = regge_wheeler, parms = params)
df <- data.frame(solution)
names(df) <- c("time", "u")
custom_theme <- theme_minimal() +
  theme(
    plot.title = element_text(size = 16, face = "bold", hjust = 0.5),
    axis.title.x = element_text(size = 14),
    axis.title.y = element_text(size = 14),
    axis.text = element_text(size = 12),
    axis.text.x = element_text(angle = 45, vjust = 0.5, hjust = 1),
    legend.title = element_blank(),
    legend.text = element_text(size = 12),
    legend.position = "bottom",
    panel.grid.major = element_blank(),
    panel.grid.minor = element_blank(),
    panel.border = element_blank(),
    panel.background = element_blank(),
    plot.background = element_rect(fill = "white"),
    strip.text = element_text(size = 14, face = "bold"),
    plot.margin = margin(1, 1, 1, 1, "cm")
  )
ggplot(df, aes(x = time, y = u, color = "Vibration")) +
  geom_line(size = 1) +
  labs(
    x = "Time",
    y = expression(u(x)),
    title = "Regge-Wheeler Black Hole Vibration",
    caption = "Source: Grafik prosesing by author"
  ) +
  scale_color_manual(values = c("#E31A1C")) +
  custom_theme
```

Figure 8. Programming algorithm for visualizing Figure 7
Source: Data processing and image generation by the author

Through the Regge-Wheeler differential equation, we can apply mathematical methods and numerical techniques to investigate black hole vibrations in the context of Regge-Wheeler. The solution to this differential equation provides information about the nature and characteristics of black hole vibrations, such as vibration frequency, energy distribution, and vibration patterns. The physical implications are a deeper understanding of black hole dynamics and the properties of its vibrations.

The problems in Table 1 are specifically designed for advanced levels of study in theoretical physics or astrophysics, with a relatively high level of difficulty. These problems involve concepts and formulas in general relativity theory and the Regge-Wheeler equation used to study perturbations in black holes. Undergraduate or graduate students in the field of physics or related disciplines studying general relativity, astrophysics, or quantum mechanics may encounter similar assignments. These problems necessitate a solid comprehension of the underlying concepts and the proficiency to apply complex formulas and advanced mathematical techniques in order to tackle physics problems effectively.

Table 1. Exercise questions on Regge-Wheeler perturbations

No	Recent exercise questions
1	The provided formula for the solution of the Regge-Wheeler perturbation in a black hole is denoted as: $\Psi(t,r,\theta,\phi) = R(r)\cdot S(\theta,\phi)e^{(-i\omega t)}$ with ω as a parameter derived from the solution formula. a. Determine the first and second derivatives of Ψ with respect to time t. b. Assuming R(r) = e(-iωr*)F(r), where r* represents the "tortoise" coordinate and F(r) is an unknown function, compute the first and second derivatives of R(r*) with respect to r*.
2	Given that the Regge-Wheeler equation for perturbations in the radial coordinate r is: $\frac{d^2R(r)}{dr^2} + \left(\frac{1}{r}\right)\frac{dR(r)}{dr} + \left[\left\{\omega^2 - V(r)\right\}R(r)\right] = 0$ where V(r) is the effective potential given by the formula: $V(r) = \left(1 - 2\frac{M}{r}\right)\left\{l\frac{(l+1)}{r^2} + 2\frac{M}{r^3}\right\}$ where M is the mass of the black hole, r is the radial distance, and l is the orbital quantum number.. a. Using the equation above, let's demonstrate how the Regge-Wheeler equation can be written in the form of the Schrödinger equation by substituting R(r) with F(r). b. Determine the first and second derivatives of V(r) with respect to r. c. If M = 2M⨀, l = 2, and ω = $3M^{(-1)}$, determine the numerical value of V(r) at r = 6M and calculate the value of ω^2 - V(r). d. Provide a physical interpretation of the value of ω^2 - V(r) at r = 6M.
3	Given that the function S(θ, ϕ) as the solution formula for the Black Hole Regge-Wheeler perturbation is : S(θ, ϕ) = P(θ)$e^{(im\varphi)}$ where P(θ) is the Legendre polynomial function and m is the magnetic quantum number. a. Provide the general form of the P(θ) function for l = 2. b. Determine the first and second derivatives of S(θ, ϕ) with respect to ϕ . c. If m = -2, determine the numerical value of S(θ, ϕ) at ϕ = π/3. d. Provide a physical interpretation of the value of S(θ, ϕ) at ϕ = π/3.
4.	Given that the Regge-Wheeler equation for perturbations in the angular coordinate θ is: $\frac{1}{sin\theta}\frac{d}{d\theta}\left\{sin\theta\cdot\frac{dP(\theta)}{d\theta}\right\} + \left\{\frac{\left(m^2 - cos^2\theta\right)}{sin^2\theta}\right\}\cdot P(\theta) = 0$ where P(θ) is the Legendre polynomial function and m is the magnetic quantum number. a. Determine the first and second derivatives of P(θ) with respect to θ using the provided equation. b. If m = 1 dan P(θ) = $3\theta^2$-2, calculate the numerical value of the first derivative and second derivative of P(θ) with respect to θ at θ = π/4. c. Provide a physical interpretation of the first derivative and second derivative values of P(θ) with respect to θ at θ = π/4.
5.	Given that the Schrödinger equation for the function F(r) in the Regge-Wheeler equation is denoted as: $d^2F(r)/dr*^2 + \left\{\omega^2 - V(r)\right\}F(r) = 0$ where V(r) is the effective potential given in the previous question. a. Using the equation above, determine the first derivative and second derivative of F(r) with respect to r*. b. If V(r) = $2M/r^3$, determine the numerical value of the first derivative and second derivative of F(r) with respect to r* at r = 3M. c. Provide a physical interpretation of the first derivative and second derivative values of F(r) with respect to r* at r = 3M.
6	Given that in Black Hole Regge-Wheeler perturbations, ω is a parameter derived from the solution formula and can be expressed in the form ω = Re(ω) + iIm(ω), where Re(ω) represents the real part of ω and Im(ω) represents the imaginary part of ω. a. If Re(ω) = 2 and Im(ω) = -3, determine the absolute value of ω. b. If $\omega^2 = 5^2 + 6^2$, determine the value of ω. c. If Re(ω) = 0 and Im(ω) = 4, determine the phase value of ω in radians. d. If Re(ω) = -1 and Im(ω) = 0, determine the absolute value of ω.
7	Given that the Regge-Wheeler equation for perturbations in time t is: $\frac{d^2h(t)}{dt^2} + \omega^2h(t) = 0$ where h(t) is the perturbation function dependent on time t, and ω is a parameter derived from the solution formula. a. If ω = 3 and h(t) = cos(ωt) + 2sin(ωt), determine the first derivative and second derivative of h(t) with respect to time t. b. If ω = 1/2 and h(t) = e^(ωt), determine the first derivative and second derivative of h(t) with respect to time t. c. If ω = -4 and h(t) = 5e^(-2ωt), determine the first derivative and second derivative of h(t) with respect to time t.
8	Given that in Black Hole Regge-Wheeler perturbations, the parameter ω is derived from the solution formula $\omega^2=V(r)+c^2k^2$, where c is the speed of light and k is the wave number. a. If V(r) = $3M/r^3$ and c = $3x10^8$ m/s, calculate the numerical value of ω at r = 2M and k = $4x10^6$ $m^{(-1)}$. b. If V(r) = $2M/r^2$ and c = $2x10^8$ m/s, calculate the numerical value of ω at r = 3M and k = $5x10^7$ $m^{(-1)}$. c. If V(r) = $4M/r^4$ and c = $2.5x10^8$ m/s, calculate the numerical value ω at r = 4M and k = $3x10^5$ $m^{(-1)}$
9	Given that in Black Hole Regge-Wheeler perturbations, the parameter k is derived from the solution formula k = m/R, where m is the magnetic quantum number and R is the radius of the orbit. a. If m = 3 and R = $5x10^3$ m, calculate the numerical value of k. b. If m = -2 and R = $2x10^4$ m, calculate the numerical value of k. c. If m = 0 and R = $1x10^5$ m, calculate the numerical value of k.

10	Given that in Black Hole Regge-Wheeler perturbations, the solution formula for the parameter ω is $\omega = c\sqrt{\left\{k^2 + \left(4\frac{M^2\omega^2}{c^4}\right)\right\}}$, where c is the speed of light, k is the wave number, and M is the mass of the black hole. a. If $c = 3x10^8$ m/s, $k = 2x10^6$ $m^{(-1)}$, dan $M = 10^5$ kg, calculate the numerical value of ω. b. If $c = 2x10^8$ m/s, $k = 5x10^7$ $m^{(-1)}$, dan $M = 5x10^6$ kg, calculate the numerical value of ω. c. If $c = 2.5x10^8$ m/s, $k = 3x10^5$ $m^{(-1)}$, dan $M = 2x10^4$ kg, calculate the numerical value of ω.

Source: Data processing and image generation by the author

This innovation is considered creative because it incorporates the context of Regge-Wheeler black hole perturbations into creating exercise questions, which can encourage students to develop a deeper and more critical understanding of the concepts involved. The goal of this innovation is to enhance student learning in theoretical physics and mathematical physics courses by providing more relevant and contextual practice questions. The benefits are twofold: for instructors, it serves as a tool to assess student competency, while for students, it enhances their understanding and skills in applying concepts of general relativity and black holes to the case of Regge-Wheeler black hole perturbations.

The Discussion

In this study, the author visualized the vibrations of a black hole using the Regge-Wheeler equation. In theoretical physics of black holes, the Regge-Wheeler equation is used to model black hole vibrations. The visualization showcased the vibrations occurring around the black hole based on the Regge-Wheeler equation.

The Results of Black Hole Vibration Visualization

The generated visualization is a two-dimensional plot displaying Cartesian coordinates (x, y) obtained from the polar coordinate transformation (r, θ). The color of the data points on the plot represents the values of the Regge-Wheeler function, with blue indicating lower values and red indicating higher values. This plot provides a visual representation of the vibrations occurring around the black hole based on the Regge-Wheeler equation.

Application of the Program

This program used the Regge-Wheeler equation to calculate the values of the Regge-Wheeler function. The function was computed based on a formula that involves several parameters and variables such as r, θ, M, ω, l, and p. These values were used to compute the Cartesian coordinates (x, y) from the polar coordinates (r, θ). The program generated point data with Cartesian coordinates and corresponding values of the Regge-Wheeler function. Subsequently, this data was used to create plots using the {ggplot2} library.

The Graph of Effective Potential

Additionally, this study also presented the effective potential graph in the Regge-Wheeler perturbation theory. The effective potential was described as a function of radial distance (r) and was a solution to the Laplace equation in curved spacetime around a black hole. This graph represents the effective potential for various parameter values, with the color of the lines indicating the variation of the parameters.

The Graph of Radial Wave Function

This study also presented visualizations of the radial wave function in the context of Regge-Wheeler vibrations around a black hole. The graph used radial coordinate (r) as the x-axis, polar angle (θ) as the y-axis, and the amplitude of the radial wave as the z-axis. The radial wave function as used to depict the Regge-Wheeler vibrations occurring around the black hole and reflects the relativistic nature of spacetime and the orbital motion of particles in a gravitational field.

The Solution of the Regge-Wheeler Differential Equation:

This study also explains the steps to obtain the exact solution of the Regge-Wheeler differential equation. The solution of this differential equation provides information about the characteristics and properties of black hole vibrations, such as vibration frequency, energy distribution, and vibration patterns. By obtaining the exact solution, researchers can gain valuable insights into the behavior and nature of vibrations around black holes, further enhancing our understanding of these gravitational phenomena.

CONCLUSION

This study used the Regge-Wheeler equation to model vibrations in black holes and generates two-dimensional graphical visualizations. The visualization demonstrates the Cartesian coordinates (x, y) resulting from the transformation of polar coordinates, where the color of the plotted points on the graph represents the values of the Regge-Wheeler function. The plot provides a visual representation of the vibrations occurring around a black hole based on the Regge-Wheeler equation. Additional visualizations in this study showcase the effective potential in the Regge-Wheeler perturbation theory and the radial wave function, depicting the vibrations in the Regge-Wheeler framework around a black hole. The analysis of the solutions of the Regge-Wheeler differential equation and the resulting visualizations provide insights into the characteristics of black hole vibrations, such as vibration frequency, energy distribution, and vibration patterns. This research has profound physical implications in understanding the dynamics of black holes, the properties of gravity, and the nature of spacetime around black holes. Additionally, this research includes exercise problems that are relevant to the theory of general relativity and the Regge-Wheeler equation. These problems are designed for undergraduate or graduate students studying physics or related fields.

Implement the visualizations generated from this research in teaching and learning physics related to black hole theory and Regge-Wheeler vibrations. These visualizations can help students and researchers to intuitively understand the concepts and properties of black hole vibrations. Develop software or computer programs that can generate similar visualizations. This will enable researchers and students to generate their own visualizations and conduct further analysis on black hole vibrations. Use the findings of this research as a reference in further studies on black holes, general relativity theory, and astrophysics. This research can provide deeper insights into the vibrations occurring around black holes and their characteristics.

Further advance this research by exploring the intricacies of black hole vibrations through the application of the Regge-Wheeler equation. Extend the analysis by incorporating additional parameters and variables to examine their influence on vibration patterns and characteristics. Employ more sophisticated mathematical techniques and numerical methods to accurately solve the Regge-Wheeler differential equation and obtain precise solutions. This comprehensive approach will enhance our understanding of the intricate nature of black hole vibrations and potentially unveil novel phenomena associated with these celestial objects. Additionally, delve deeper into the physical implications of the research findings, investigating the impact of black hole vibrations on their structural properties and the gravitational effects on wave phenomena. This research can contribute to a deeper understanding of general relativity, astrophysics, and the nature of spacetime surrounding black holes. It can serve as a stepping stone for further investigations into black hole vibrations within broader areas of physics, such as quantum mechanics and quantum gravity. This has the potential to enhance our comprehension of the fundamental characteristics of the universe and their connections to black holes.

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