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The Development of New Operations for Tensor Products of Product Fuzzy Graphs, Strong Product Fuzzy Graphs, and Complete Product Fuzzy Graphs

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ABSTRACT

The purpose of this study is to develop new properties of product fuzzy graphs in the form of tensor product operations. The research method used consists of a preliminary stage, a stage of defining and theorems, a stage of proving theorems and verifying research results. The results of this study obtained the definition of tensor product operations from fuzzy graph products. Furthermore, it was found that the graph resulting from tensor product operations is a fuzzy graph, the tensor product operations of strong product fuzzy graphs are strong, and the tensor product operations of complete product fuzzy graphs are strong.

Keywords: complete graph, fuzzy graph, product fuzzy graph, strong graph, tensor product

INTRODUCTION

Research related to fuzzy graphs is more focused on the formation of new definitions and theorems that lead to their use in solving real-world problems. Josy et al. (2022) applied fuzzy graphs to solve social problems in society (Josy et al., 2022). Farhang and Telebi (2024) used fuzzy graphs to solve dynamic process problems. (Farhang & Talebi, 2024).

Other relevant research related to fuzzy graph applications can be found in (Zhu & Xu, 2016), (Akram et al., 2018), (Mohamed & Ali, 2021), (Bharathi & Leo, 2023), and (Mahalakshmi, 2024). In addition to developing in terms of application, research related to fuzzy graphs has also developed in terms of theory.

Ramaswamy and Poornima in 2009 introduced the concept of a fuzzy product graph by replacing the minimum operation in the definition of a fuzzy graph with a product operation (Ramaswamy & Poornima, 2009). In 2012, Firmansah and Bayu proved that the complement of the multiplication of complete product fuzzy graphs of their complements, where this configuration produces a null graph (Firmansah & Bayu, 2012).

Al-Hawary and Hourani in 2016 introduced new operations on product fuzzy graphs (Al-Hawary & Hourani, 2016). Akram et al. (2020) introduced the concept of product residue from fuzzy graph structure (Akram et al., 2020). Javaid et al. (2020) introduced hesitant fuzzy graphs along with the applicable operations. (Javaid et al., 2020).

Muhiuddin et al in 2022 introduced the concept of independent fuzzy graphs (Muhiuddin et al., 2022). Shubatah & Haifa in 2023 introduced the concept of global dominance numbers and global dominance numbers in fuzzy graph products (Shubatah & Haifa, 2023).

Firmansah et al. in 2025 defined the cartesian product operation on product fuzzy graphs, strong product fuzzy graphs, and complete product fuzzy graphs (Firmansah et al., 2025). In line with this research, there is still a research gap in the form of other product operations of product fuzzy graphs that have not been defined. Therefore, this paper will develop a new operation in the form of a tensor product of product fuzzy graphs. The novelty obtained in this study is the tensor product operation of product fuzzy graphs.

MATERIAL AND METHOD

This research is a qualitative development study. The research stages consist of a preliminary stage to identify open issues from relevant studies. Next is the definition construction stage, which involves forming a new graph class definition with its properties. At this stage, a theorem stating the properties of the constructed graph definition will also be formed. This is followed by the theorem proof and research result verification stage, which involves proving the mathematically formed theorem.

RESULT AND DISCUSSION

The following are definitions of fuzzy graphs in Definition 1, product fuzzy graphs in Definition 2, complete product fuzzy graphs in Definition 3, and strong product fuzzy graphs in Definition 4.

Definition 1. (Rosenfeld, 1975)
Let $G(\alpha,\beta)$ with $G^*(V,E)$ is a fuzzy graph then fulfill

$$\beta(xy) \leq \alpha(x) \wedge \alpha(y)$$

For all $x, y \in V$.

Definition 2. (Ramaswamy & Poornima, 2009)
Let $G(\alpha,\beta)$ with $G^*(V,E)$ is a product fuzzy graph then fulfill

$$\beta(xy) \leq \alpha(x)\alpha(y)$$

For all $x, y \in V$

Definition 3. (Ramaswamy & Poornima, 2009)
Let $G(\alpha,\beta)$ with $G^*(V,E)$ is a complete product fuzzy graph then fulfill

$$\beta(xy) = \alpha(x)\alpha(y)$$

For all $x, y \in V$.

Definition 4. (Ramaswamy & Poornima, 2009)
Let $G(\alpha,\beta)$ with $G^*(V,E)$ is a strong product fuzzy graph then fulfill

$$\beta(xy) = \alpha(x)\alpha(y)$$

For all $xy \in E$.

The following will provide the definition of the tensor product of fuzzy graphs in Definition 5 and the definition of the tensor product of product fuzzy graphs in Definition 6.

Definition 5. (Dogra, 2015)
Tensor product of fuzzy graphs $G_1:(\alpha_1,\beta_1)$ with $G_1^*:(V_1,E_1)$ and $G_2:(\alpha_2,\beta_2)$ with $G_2^*:(V_2,E_2)$ by assumption $V_1 \cap V_2 \neq \emptyset$ defined as a fuzzy graph $G_1 \otimes G_2 : (\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ with $G^*:(V_1 \otimes V_2, E_1 \otimes E_2)$ with
$V_1 \otimes V_2 = \{(u_1,v_1) | u_1 \in V_1, v_1 \in V_2\}$
and
$E_1 \otimes E_2 =$
$\{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$
with
$(\alpha_1 \otimes \alpha_2)(u_1,v_1) = \alpha_1(u_1) \wedge \alpha_2(v_1)$
for all $(u_1,v_1) \in V_1 \otimes V_2$
$(\beta_1 \otimes \beta_2)\big((u_1,v_1)(u_2,v_2)\big) =$
$\beta_1(u_1u_2) \wedge \beta_2(v_1v_2)$,
if $u_1u_2 \in E_1, v_1v_2 \in E_2$.

Based on Definition 5, the tensor product operation is developed on the product fuzzy graph stated in Definition 6.

Definition 6.
Tensor product of product fuzzy graphs $G_1:(\alpha_1,\beta_1)$ with $G_1^*:(V_1,E_1)$ and $G_2:(\alpha_2,\beta_2)$ with $G_2^*:(V_2,E_2)$ by asumption $V_1 \cap V_2 \neq \emptyset$ defined as a fuzzy graph
$G_1 \otimes G_2 : (\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ with $G^*:(V_1 \otimes V_2, E_1 \otimes E_2)$ with

$V_1 \otimes V_2 = \{(u_1, v_1) | u_1 \in V_1, v_1 \in V_2\}$
and
$E_1 \otimes E_2 = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$
with
$(\alpha_1 \otimes \alpha_2)(u_1, v_1) = \alpha_1(u_1) \times \alpha_2(v_1)$
for all $(u_1, v_1) \in V_1 \otimes V_2$
$(\beta_1 \otimes \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \beta_1(u_1u_2) \times \beta_2(v_1v_2)$,
if $u_1u_2 \in E_1, v_1v_2 \in E_2$.

Example 1.
For example, given product fuzzy graph G_1 and product fuzzy graph G_2 in Figure 1, then the tensor product of the product fuzzy graph $G_1 \otimes G_2$ is obtained as follows.

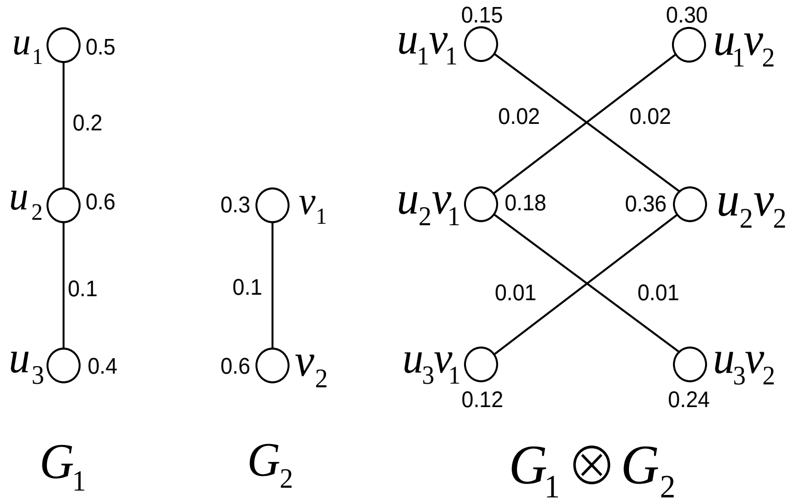

Figure 1. Tensor product of the product fuzzy graph $G_1 \otimes G_2$.

Next, several properties satisfied by the tensor product operation of the product fuzzy graph are given, as stated in Theorem 1, Theorem 2, and Corollary 3.

Theorem 1.
Let $G_1(\alpha_1, \beta_1)$ with $G_1^*(V_1, E_1)$ and $G_2(\alpha_2, \beta_2)$ with $G_2^*(V_2, E_2)$ are product fuzzy graph then tensor product of the product fuzzy graph $G_1 \otimes G_2 : (\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ is a fuzzy graph.

Proof.
Let $G_1(\alpha_1, \beta_1)$ with $G_1^*(V_1, E_1)$ is a product fuzzy graph then fulfill

$$\beta_1(u_1u_2) \le \alpha_1(u_1) \times \alpha_1(u_2) \le \alpha_1(u_1) \wedge \alpha_1(u_2) \quad (1)$$

and $G_2(\alpha_2, \beta_2)$ with $G_2^*(V_2, E_2)$ is a product fuzzy graph then fulfill

$$\beta_2(v_1v_2) \le \alpha_2(v_1) \times \alpha_2(v_2) \le \alpha_2(v_1) \wedge \alpha_2(v_2) \quad (2)$$

Based on (1) and (2) obtained
For $u_1u_2 \in E_1, v_1v_2 \in E_2$ then fulfill
$(\beta_1 \times \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \beta_1(u_1u_2) \times \beta_2(v_1v_2)$
$= \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$\le (\alpha_1 \times \alpha_2)(u_1, v_1) \wedge (\alpha_1 \times \alpha_2)(u_2, v_2)$
Obtained $(\beta_1 \times \beta_2)\big((u_1, v_1)(u_2, v_2)\big) \le (\alpha_1 \times \alpha_2)(u_1, v_1) \wedge (\alpha_1 \times \alpha_2)(u_2, v_2)$ so that the tensor product of the product fuzzy graph product $G_1 \otimes G_2 : (\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ is a fuzzy graph.

Theorem 2.
Let $G_1(\alpha_1, \beta_1)$ with $G_1^*(V_1, E_1)$ and $G_2(\alpha_2, \beta_2)$ with $G_2^*(V_2, E_2)$ are strong product fuzzy graph then tensor product of the product fuzzy graph $G_1 \otimes G_2 : (\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ is a strong.

Proof.
Let $G_1(\alpha_1, \beta_1)$ with $G_1^*(V_1, E_1)$ is a strong product fuzzy graph then fulfill

$\beta_1(u_1u_2) = \alpha_1(u_1) \times \alpha_1(u_2) =$
$s\alpha_1(u_1) \wedge \alpha_1(u_2)$ (1)

and $G_2(\alpha_2, \beta_2)$ with $G_2^*(V_2, E_2)$ is a strong product fuzzy graph then fulfill

$\beta_2(v_1v_2) = \alpha_2(v_1) \times \alpha_2(v_2) =$
$\alpha_2(v_1) \wedge \alpha_2(v_2)$ (2)

Based on (1) and (2) obtained
For $u_1u_2 \in E_1, v_1v_2 \in E_2$ then fulfill
$(\beta_1 \times \beta_2)\big((u_1, v_1)(u_2, v_2)\big) =$
$\beta_1(u_1u_2) \times \beta_2(v_1v_2)$

$= \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$

$= (\alpha_1 \times \alpha_2)(u_1, v_1) \wedge (\alpha_1 \times \alpha_2)(u_2, v_2)$

Obtained $(\beta_1 \times \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = (\alpha_1 \times \alpha_2)(u_1, v_1) \wedge (\alpha_1 \times \alpha_2)(u_2, v_2)$
the tensor product of the strong product fuzzy graph $G_1 \otimes G_2 : (\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ is a strong.

Corollary 3.
Let $G_1(\alpha_1, \beta_1)$ with $G_1^*(V_1, E_1)$ and $G_2(\alpha_2, \beta_2)$ with $G_2^*(V_2, E_2)$ are complete product fuzzy graph then tensor product of the product fuzzy graph $G_1 \otimes G_2 : (\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ is a strong.

Example 2.
For example, given strongs product fuzzy graph G_1 and strong sproduct fuzzy graph G_2 in Figure 2, then the tensor product of the strong product fuzzy graph $G_1 \otimes G_2$ is obtained as follows.

Example 3.
Figure 2 also shows that the complete fuzzy product graph is a strong fuzzy product graph.

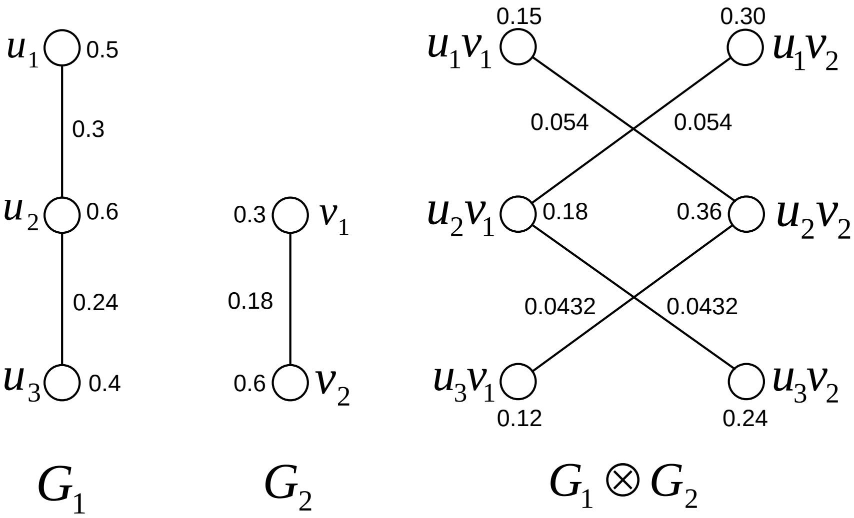

Figure 2. Tensor product of the strong product fuzzy graph $G_1 \otimes G_2$.

Based on the above research results, it is obtained that the tensor product of a fuzzy product graph is a fuzzy graph (Theorem 1), the tensor product of a strong product fuzzy graphs are strong (Theorem 2), and the tensor product of a complete product fuzzy graph are strong (Corollary 3). These results are in line with the research of Firmansah et al in 2025 (Firmansah et al., 2025).

CONCLUSION

The results of the study obtained that the tensor product of a fuzzy product graph is a fuzzy graph, the tensor product of a strong product fuzzy graphs are strong and the tensor product of a complete product fuzzy graph are strong

The suggestion for further research is to find other definitions of operations on product fuzzy graphs, such as normal product and modular product operations. Furthermore, this research can also be developed for other classes of graphs.

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